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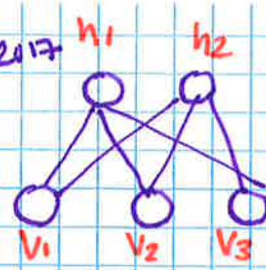

$$\boxed{-E(v,h) = \sum_i \sum_j h_i W_{ij} v_j}$$

$$\begin{cases} h_i W_{ij} v_j = 0 & \text{if } h_i = 0 \text{ or } v_j = 0 \\ h_i W_{ij} v_j = 1 & \text{when } h_i = 1 \text{ and } v_j = 1 \end{cases}$$

	h		v			$-E(v,h)$
	h_1	h_2	v_1	v_2	v_3	
	1	0	0	0	0	0
	1	0	0	0	1	W_{13}
A	1	0	0	1	0	W_{12}
	1	0	0	1	1	$W_{12} + W_{13}$
	1	0	1	0	0	W_{11}
	1	0	1	0	1	$W_{11} + W_{13}$
	1	0	1	1	0	$W_{11} + W_{12}$
	1	0	1	1	1	$W_{11} + W_{12} + W_{13}$
	1	1	0	0	0	0
	1	1	0	0	1	$W_{13} + W_{23}$
B	1	1	0	1	0	$W_{12} + W_{22}$
	1	1	0	1	1	$W_{12} + W_{13} + W_{22} + W_{23}$
	1	1	1	0	0	$W_{11} + W_{21}$
	1	1	1	0	1	$W_{11} + W_{13} + W_{21} + W_{23}$
	1	1	1	1	0	$W_{11} + W_{12} + W_{21} + W_{22}$
	1	1	1	1	1	$W_{11} + W_{12} + W_{13} + W_{21} + W_{22} + W_{23}$
	0	1	0	0	0	0
	0	1	0	0	1	W_{23}
C	0	1	0	1	0	W_{22}
	0	1	0	1	1	$W_{22} + W_{23}$
	0	1	1	0	0	W_{21}
	0	1	1	0	1	$W_{21} + W_{23}$
	0	1	1	1	0	$W_{21} + W_{22}$
	0	1	1	1	1	$W_{21} + W_{22} + W_{23}$
	0	0	0	0	0	0
	0	0	0	0	1	0
	0	0	0	1	0	0
D	0	0	0	1	1	0
	0	0	1	0	0	0
	0	0	1	0	1	0
	0	0	1	1	0	0
	0	0	1	1	1	0

(A) W_{1j}

$$\prod_j \left(1+e^{W_{1j}}\right) = \left(1+e^{W_{11}}\right)\left(1+e^{W_{12}}\right)\left(1+e^{W_{13}}\right)$$

$$= \left(1+e^{W_{11}}+e^{W_{12}}+e^{W_{11}+W_{12}}\right)\left(1+e^{W_{13}}\right)$$

$$= 1+e^{W_{11}}+e^{W_{12}}+e^{W_{11}+W_{12}}+e^{W_{11}+W_{13}}+e^{W_{12}+W_{13}}+e^{W_{11}+W_{12}+W_{13}}$$

$$= e^{-E([0,0,0],A)} + e^{-E([0,1,0],A)} + e^{-E([1,1,0],A)} + \dots + e^{-E([1,1,1],A)}$$

so this expand to all possible summations for $h_1 = 1$ $h_2 = 0$
(you can check with the table from previous page)

(B) $W_{1j} + W_{2j}$

$$\prod_j \left(1+e^{W_{1j}+W_{2j}}\right) = \left(1+e^{W_{11}+W_{21}}\right)\left(1+e^{W_{12}+W_{22}}\right)\left(1+e^{W_{13}+W_{23}}\right) =$$

$$\left(1+e^{W_{11}+W_{21}}+e^{W_{12}+W_{22}}+e^{W_{11}+W_{21}+W_{12}+W_{22}}\right)\left(1+e^{W_{13}+W_{23}}\right) =$$

$$1+e^{W_{11}+W_{21}}+e^{W_{12}+W_{22}}+e^{W_{11}+W_{21}+W_{12}+W_{22}}+e^{W_{11}+W_{21}+W_{13}+W_{23}}+e^{W_{12}+W_{22}+W_{13}+W_{23}}+e^{W_{11}+W_{21}+W_{12}+W_{22}+W_{13}+W_{23}}$$

$$= e^{-E([0,0,0],B)} + e^{-E([1,0,0],B)} + e^{-E([0,1,0],B)} + \dots + e^{-E([1,1,1],B)}$$

Same here, the product expands to all possible cases when
$h_1 = 1$, $h_2 = 1$

(C) W_{2j}

$$\prod_j \left(1+e^{W_{2j}}\right) = \text{same idea } \left(1+e^{W_{21}}\right)\left(1+e^{W_{22}}\right)\left(1+e^{W_{23}}\right)$$

$$e^{E([0,0,0],C)} + e^{-E([1,0,0],C)} + \dots + e^{-E([1,1,1],C)}$$

D Since all $E(v, D) = 0$

then ← size of

$$\sum_{v} e^{E(v,D)} = 2^{\|v\|} = 8$$

or in other words

$$\underbrace{e^0 + e^0 + \cdots + e^0}_{8 \text{ times}}.$$

Now we need an algorithm to compute all these W combinations.
for that we can take advantage of the quadratic form:

$$hWv,$$

Using h and v to "select" the summations over W that we need
this is what I came with:

A $\cancel{[1, 0]}$ $\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then we can

$$[W_{11} \; W_{12} \; W_{13}] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = W_{11} \longrightarrow (1 + e^{W_{11}})$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = W_{12} \longrightarrow (1 + e^{W_{12}})$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = W_{13} \longrightarrow (1 + e^{W_{13}})$$

and so we have what we need for A

(B) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} W_{11}+W_{21} & W_{12}+W_{22} & W_{13}+W_{23} \end{bmatrix}$

And so we have

$(1+e^{W_{11}+W_{21}})(1+e^{W_{12}+W_{22}})(1+e^{W_{13}+W_{23}})$

which is what we need for (B)!

(C) Same logic applies here: $h_1=0 \quad h_2=1$

$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \end{bmatrix} I$

$\begin{bmatrix} W_{21} & W_{22} & W_{23} \end{bmatrix}$

$(1+e^{W_{21}})(1+e^{W_{22}})(1+e^{W_{23}})$

So going back to the original problem of calculating the Z partition function, we can divided into a summation of cases

(A) + (B) + (C) + (D) $\qquad Z = \sum_{v,h} e^{-E(v,h)}$

then we have a way to compute the total energy for each possible configuration of h by expressing it as products over all possible configurations of v.

$$Z = \underbrace{2^{\|v\|}}_{D} + \underbrace{\prod_j (1+e^{W_{1j}})}_{A} + \underbrace{\prod_j (1+e^{W_{2j}})}_{C} + \underbrace{\prod_j (1+e^{W_{1j}+W_{2j}})}_{B}$$

In the general case we will have as many products as possible h configurations but that is ok since we can use the quadratic form to make the appropriate selections for computation → now to write the code! ... and calculate

ln Z